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GRAVITATIONAL STABILITY OF A MASSLESS SCALAR FIELD

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A solution is obtained to the Einstein equations for a static, spherically symmetric, massless scalar field. The stability of a static, massless scalar field for a point source is studied. It is found that the field is unstable for small ratios of the scalar charge of the source to its mass but is stable for large ratios. It is proven that there exists a dimensionless number which limits the region of stability for the field.

The problem of gravitational stability of a static, spherically symmetric, massless scalar field was formulated in [1], where the authors concluded that the scalar field of isolated sources is unstable. We will show that this conclusion is unfounded and that there exists a region of stability when the ratio of the charge to the mass of a point source exceeds some fully specified quantity (there is a dimensionless parameter that enters into the static solution which limits the regions of stability and instability).

We have for the metric

$$ds^2 = -e^{\alpha}dr^2 - e^{\beta}(d\theta^2+\sin^2\theta d\varphi^2) + e^{\gamma}c^2dt^2, \tag{1}$$

where α, β, and γ are functions of the radial coordinate r and the time coordinate $x = ct$, and the equation for the massless scalar field will be

$$\psi'' - \frac{1}{2}(\alpha' - 2\beta' - \gamma')\psi' - e^{\alpha-\gamma}\left[\ddot{\psi} + \frac{1}{2}(\dot{\alpha} + 2\dot{\beta} - \dot{\gamma})\dot{\psi}\right] = 0. \tag{2}$$

We will assume that

$$\alpha = \alpha_0 + \delta\alpha,\ \beta = \beta_0 + \delta\beta,\ \gamma = \gamma_0 + \delta\gamma,\ \psi = \psi_0 + \delta\psi, \tag{3}$$

where α_0, β_0, γ_0, and ψ_0 are the solution of the Einstein equations for static fields of a point source, and $\delta\alpha$, $\delta\beta$, $\delta\gamma$, and $\delta\psi$ are small perturbations that are functions of time. We will obtain the equation for $\delta\psi$, leaving the form of the static solution indeterminate (in contrast to [1]). We will select the perturbation coordinates so that $\delta\beta = 0$. Substituting (3) into (2) gives the equation for $\delta\psi$, which contains the quantity $(\delta\alpha' - \delta\gamma')$. We will express this quantity in terms of $\delta\psi$. We obtain the following from Einstein's equations for metric (1) with the energy-momentum tensor of a massless scalar field (assuming that $\beta = \beta_0$ and is not a function of time)

$$\delta\alpha = \frac{4G}{c^4}\frac{\psi_0'}{\beta_0'}\delta\psi \tag{4}$$

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and

$$\frac{1}{2}\beta'(\alpha'-\gamma')=\beta''+\beta'^2-2e^{\alpha-\beta}. \tag{5}$$

Describing the latter equality first for the static solution and then for the "perturbed" solution while using (4), we can obtain the quantity ($\delta\alpha'-\delta\gamma'$), after which we will find the equation for a small perturbation in the potential of a massless scalar field in the form

$$\delta\psi''-\frac{1}{2}\varphi(r)\,\delta\psi'+f(r)\,\delta\psi-g(r)\,\delta\ddot{\psi}, \tag{6}$$

where

$$f(r)=\frac{8G}{c^4}\left(\frac{\psi_0'}{\beta_0'}\right)^2\cdot e^{\alpha_0-\beta_0}, \tag{7}$$

$$g(r)=e^{\alpha_0-\gamma_0}, \tag{8}$$

$$\varphi(r)=\alpha_0'-2\beta_0'-\gamma_0'. \tag{9}$$

Separating variables, we have

$$\delta\psi=\delta\psi_0(x^0)\,\delta\psi_1(r),\quad \delta\psi_0=\exp(\pm i\omega x^0), \tag{10}$$

$$\delta\psi_1''-\frac{1}{2}\varphi(r)\,\delta\psi_1'+(\omega^2 g+f)\,\delta\psi_1=0. \tag{11}$$

The separation constant (ω^2) determines the circular frequency (ωc) of the perturbed radial "pulsations." If (11) has negative eigenvalues ω^2, then, according to (10), it is possible to have an unlimited increase of $\delta\psi_0$ with time, which is also true for $\delta\psi$, i.e., the scalar field in this case is unstable.

The actual form of Eq. (11) depends on the form of the static solution. The form of the latter is determined by the selection of the coordinate conditions for which the static Einstein equations are solved.

We will select the following relation for the coordinate condition

$$\alpha_0-2\beta_0-\gamma_0=-4\ln r. \tag{12}$$

Then, the solution of the static equations in the Newtonian limit will be

$$\beta_0=(\kappa+1)\frac{r_g}{r}+2\ln\left\{\kappa r_g\Big/\left[\exp\left(\frac{\kappa r_g}{r}\right)-1\right]\right\}, \tag{13}$$

$$\psi_0=-q/r,\ \gamma_0=-r_g/r,$$

where r_g is the gravitational radius of the source, q is its scalar charge, and

$$\kappa=+\sqrt{1+\frac{1}{G}\left(\frac{q}{m}\right)^2}. \tag{14}$$

In this case

$$f(r)=\frac{8Gq^2}{c^4r^8}\cdot\frac{1}{\beta_0'^2}\cdot e^{\beta_0+\gamma_0},\ g(r)=\frac{e^{2\beta_0}}{r^4},\ \varphi(r)=-\frac{4}{r}. \tag{15}$$

Introducing the new function

$$y(r)=r\delta\psi_1,$$

we obtain the equation

$$y''+(\omega^2 g+f)\,y=0. \tag{16}$$

Assigning homogeneous boundary conditions at the origin and at infinity, we will obtain an eigenvalue problem whose solution answers the question of the field's gravitational stability.

Equation (16) was obtained by the methods indicated in [1], where the stability of the static solutions of the Einstein equations is investigated for small perturbations. However, it differs from the corresponding equation in [1] because a different form of the static solution was used (it can be obtained from the Bronnikov solution [2] through a coordinate transformation). One should note that the Bronnikov solution has many advantages over the Fisher solution because the components of the metric tensor and the potential of the scalar field are expressed by direct functions of an independent variable).

The eigenvalue problem is also formulated differently. Since Eq. (16) possesses no singularities, the boundary conditions are first order, which requires that the function y(r) go to zero at the origin and at infinity. In [1], the equation was first transformed into the normal Liouville form. This transformation of Eq. (16) is completely unnecessary for our case, since it "degrades" the equation, giving rise to singularities and making it difficult to formulate single-valued boundary conditions in simple form. This contrast in the formulation of the eigenvalue problem also leads to results in our study and in [1] that differ significantly.

We will estimate the boundaries in which the quantity of the smallest eigenvalue ω^2 lies using the inclusion theorem in [3]. Using homogeneous boundary conditions for Eq. (16), the inclusion theorem confirms that the smallest eigenvalue lies between the minimum and the maximum of the auxiliary function

$$\Phi(r) = \frac{-u'' - fu}{gu}, \tag{17}$$

where u is an arbitrary comparison function (positive over the entire infinite interval). Using the dimensionless variable

$$x = r/r_g \tag{18}$$

we can preserve the form of Eq. (16), but instead of the quantities ω and f, we have

$$\left.\begin{aligned} \Omega &= \omega \cdot r_g, \\ h &= f \cdot r_g. \end{aligned}\right\} \tag{19}$$

Using the static solution (13) gives the equations

$$g(x) = \left(\frac{\kappa/x}{e^{\kappa/x} - 1}\right)^4 \cdot \exp\left[\frac{2(\kappa+1)}{x}\right], \tag{20}$$

$$h(x) = 2\kappa^2(\kappa^2 - 1)\frac{1}{x^4} \cdot \frac{e^{\kappa/x}}{[(\kappa-1)e^{\kappa/x} + \kappa + 1]^2}. \tag{21}$$

From here, it is evident that at the origin (for x = 0), both g and h are equal to zero. When $x = \infty$, the function h(x) is again equal to zero, but g(x) is equal to unity. Neither functions possess any singularities.

We will select the following functions for the comparison functions

$$u_1 = \frac{x}{1+x^2}, \quad u_2 = \frac{1}{x}\exp\left(-\frac{1}{x}\right). \tag{22}$$

The function $\Phi_1(x)$, which is obtained after substituting into (17) u_1 and expressions (20) and (21), determines the lower limit of the smallest eigenvalue, and the function corresponds to $\Phi_1(x)$ and gives the upper limit. Values for the functions $\Phi_1(x)$, $\Phi_2(x)$, and h(x) were tabulated for different values of the parameter κ using a computer. It was found that when κ changes from $1 + 10^{-24}$ to 2.0, the lower limit of the smallest eigen value varies from −0.4806 to −0.5514, and the upper limit varies from 0.2949 to 0.9445. For large values of the parameter κ, the lower limit remains negative, and the upper limit is positive. Hence, although it gives a representation of the order of the smallest eigenvalue, the inclusion theorem does not allow one to conclude whether the eigenvalues for different values of the parameter κ are negative or positive. One must use the Rayleigh principle for obtaining the answer to this question. In our case, the Rayleigh ratio will be [3]

$$R[u]=\frac{\int_0^\infty u'^2dx - I(\kappa)}{\int_0^\infty g(x)\,u^2dx},\quad I(\kappa)\equiv\int_0^\infty h(x)\,u^2dx.$$

The functions

$$u_n=\frac{1}{\sqrt{(n+1)(n+2)}}\,xe^{-\frac{x}{2}}L_n^2(x),$$

where $L_n^2(x)$ are the generalized Chebyshev—Laguerre polynomials, form an orthonormal system of functions over an infinite interval $(0, \infty)$ [4]. Assuming that the estimate for the first eigenvalue using the Rayleigh principle coincides with the null approximation of the Galerkin method, where the latter gives more reliable results when using orthonormal functions, we will select the first function of the indicated system as the comparison function in the Rayleigh principle, i.e. the function

$$u(x)=\frac{1}{\sqrt{2}}\,x\exp\left(-\frac{x}{2}\right).$$

The first integral in the numerator of the Rayleigh ratio can then be calculated analytically fairly easily. It is equal to 1/4. The second integral (I) for different values of the parameter κ was determined by the Gaussian method on a computer with an accuracy of 10^{-6}. For a change in κ from $1 + 10^{-6}$ to 100, the quantity I varies from 0.932 to $1.2\cdot10^{-6}$.

The decrease in the quantity of the integral I with an increase in the parameter κ is due to the character of the dependence of the function h on κ. Calculations shown that when κ is close to unity, the function h(x) has a narrow, high peak near the origin. When κ is increased, this peak is displaced in the direction of increasing x, where it becomes broader and its height is less. For large κ, the peak disappears, and the function h(x) is nearly zero over the entire infinite interval, since the integral I is also small for other test functions. It was found that for small κ, the Rayleigh ratio is negative, while for large values it is positive. Therefore, negative eigenvalues correspond to small κ, and positive eigenvalues correspond to large values. The quantity κ, for which the eigenvalues changed sign, i.e., a transition from instability to stability, was only found for the test function indicated above. It was established that this transition occurs when $\kappa \sim 4$, where the numerator of the Rayleigh ratio is equal to zero (I = 0.25).

It was found, then, that there is a dimensionless number which limits the region of stability for a massless scalar field. The role of this number in the problem of the gravitational stability of a massless scalar field is similar to, for example, the Reynolds number in hydrodynamics.

The instability of a massless scalar field of a point source when the ratio of the charge of the source to its mass is small indicates that a scalar charge that is too small cannot destroy the horizon in the metric, since the field is unstable and disappears.

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